M321 HB

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Mathematics: A Third Level Course

HANDBOOK

Partial Differential Equations of Applied Mathematics

Prepared by the Course Team

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Note

Cross-references in this handbook are indicated by the use of CAPITALS. In addition, references are given to appearances in the course of definitions/formulas/theorems. The set books are denoted by $\underset{\sim}{S}$ and $\underset{\sim}{W}$, and the correspondence text of Unit n is indicated by $\underset{\sim}{n}$.

1 GLOSSARY

ABSOLUTELY CONVERGENT

6: page 13

14: page 7

The series $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

AMPLIFICATION MATRIX

15: page 7

The THREE-LEVEL SCHEME $A\underset{\sim}{u}_{j+1} = B\underset{\sim}{u}_j + C\underset{\sim}{u}_{j-1}$ is equivalent to a pair of two-level schemes which may be written in the form

$$\begin{bmatrix} u_{j+1} \\ \\ v_{j+1} \end{bmatrix} = P \begin{bmatrix} u_j \\ \\ v_j \end{bmatrix}$$

where

$$P = \begin{bmatrix} A^{-1}B & A^{-1}C \\ \\ I & 0 \end{bmatrix}$$

P is called the <u>amplification matrix</u>.

ANGULAR FREQUENCY

7: page 12

One-dimensional wave motion which is described by $A\sin(kx-\omega t)$, for example, where x represents the displacement and t the time, has <u>angular frequency</u> ω, <u>wave length</u> $2\pi/k$ and <u>wave number</u> k.

ANGULAR MOMENTUM

16: page 13

The <u>angular momentum</u> about the point P of a particle at the point Q with velocity $\underset{\sim}{v}$ and mass m is

$$m\underset{\sim}{r} \times \underset{\sim}{v}, \text{ where } \underset{\sim}{r} = \overrightarrow{PQ}$$

ANGULAR VELOCITY

16: page 12

The <u>angular velocity</u> about the point P of a particle at the point Q with velocity $\underset{\sim}{v}$ is $(\underset{\sim}{r}\times\underset{\sim}{v})/r^2$, where $\underset{\sim}{r} = \overrightarrow{PQ}$; it measures the rate of rotation of the particle about Q.

ASYMPTOTIC RATE OF CONVERGENCE
11: page 20

The asymptotic rate of convergence of the ITERATIVE SCHEME

$$\underset{\sim}{x}^{(n+1)} = G\underset{\sim}{x}^{(n)} + H\underset{\sim}{b}$$

is $-\log \rho$, where ρ is the SPECTRAL RADIUS of G.

BAND MATRIX
11: page 16

A band matrix is one in which all the nonzero entries are confined to the leading diagonal and those diagonals close to it. If a_{ij} is that element of a matrix A which lies in row i and column j, then A is a band matrix of band width 2k+1 provided

$$a_{ij} = 0 \text{ for } |i-j| > k.$$

BAND WIDTH

See BAND MATRIX.

BESSEL'S EQUATION
14: page 6
W : page 179

Bessel's equation of order m is

$$t^2\frac{d^2u}{dt^2} + t\frac{du}{dt} + (t^2 - m^2)u = 0$$

or

$$\frac{d}{dt}\left(t\frac{du}{dt}\right) - \frac{m^2}{t}u + tu = 0.$$

BESSEL FUNCTION
W : page 179

A solution of BESSEL'S EQUATION (which is bounded at the origin) is called a Bessel function (of the first kind). A Bessel function of the first kind of order m (>0) is given by

$$J_m(t) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{1}{2}t)^{m+2k}}{k!\Gamma(m+k+1)} .$$

BODY FORCE

$\underset{\sim}{1}$: page 10

A body force is a force which acts on each element of a body from outside the body (e.g., gravitational force).

BOUNDARY CONDITIONS

$\underset{\sim}{1}$: page 9

Boundary conditions are conditions at a spatial boundary for all times. For example, the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad x \in (\alpha,\beta), \quad t > 0,$$

may be given with the boundary conditions

$$u(\alpha,t) = f(t) \qquad t \geq 0,$$
$$\frac{\partial u}{\partial x}(\beta,t) = g(t) \qquad t \geq 0.$$

BOUNDARY VALUE PROBLEM

See GREEN'S FUNCTION.

CANONICAL FORM

See STANDARD FORM.

CAUCHY PROBLEM

$\underset{\sim}{2}$: page 18

$\underset{\sim}{5}$: page 5

A Cauchy problem (or initial-boundary value problem) is a problem where the solution to a partial differential equation is required subject to the solution and its normal derivative both being specified on the initial line.

CENTRAL-DIFFERENCE OPERATOR

$\underset{\sim}{5}$: page 9

The central-difference operator δ_h is defined by

$$\delta_h u : x \longmapsto u(x+\tfrac{1}{2}h) - u(x-\tfrac{1}{2}h) \qquad x \in R.$$

CHARACTERISTICS

See CLASSIFICATION OF OPERATORS.

CLASSIFICATION OF OPERATORS

2: page 19

W: page 43

The operator

$$L:u \to A(x,t)\frac{\partial^2 u}{\partial t^2} + B(x,t)\frac{\partial^2 u}{\partial x \partial t} + C(x,t)\frac{\partial^2 u}{\partial x^2} + F\left(x,t,u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x}\right) \text{ is}$$

(a) <u>hyperbolic</u> at $(\bar{x},\bar{t})$ if $[B(\bar{x},\bar{t})]^2 - 4A(\bar{x},\bar{t})C(\bar{x},\bar{t}) > 0$,

(b) <u>parabolic</u> at $(\bar{x},\bar{t})$ if $[B(\bar{x},\bar{t})]^2 - 4A(\bar{x},\bar{t})C(\bar{x},\bar{t}) = 0$,

(c) <u>elliptic</u> at $(\bar{x},\bar{t})$ if $[B(\bar{x},\bar{t})]^2 - 4A(\bar{x},\bar{t})C(\bar{x},\bar{t}) < 0$.

For hyperbolic equations, the two families of curves which satisfy the equation

$$A\left(\frac{dx}{dt}\right)^2 - B\frac{dx}{dt} + C = 0$$

are called <u>characteristics</u>.

COEFFICIENT OF VISCOSITY

See VISCOSITY.

COMPATIBLE

See CONSISTENT.

COMPLETE

See CONVERGENCE IN THE MEAN.

COMPLEX FOURIER SERIES

6: page 18

The FOURIER SERIES expansion of a function (with domain R or C) in terms of the ORTHOGONAL BASIS

$$\{e^{inx} : n \in Z,\ x \in [-\pi,\pi]\}$$

is a <u>complex Fourier series</u>.

CONDITIONALLY STABLE

8: page 17

A FINITE-DIFFERENCE SCHEME is <u>conditionally stable</u> if it is STABLE under certain conditions (e.g., for particular values of the MESH RATIO) and unstable under others.

CONSISTENT

8: page 9

A FINITE-DIFFERENCE REPLACEMENT of a differential equation is <u>consistent</u> (or <u>compatible</u>) if its LOCAL TRUNCATION ERROR approaches zero as the MESH spacings tend to zero. (See also LAX'S THEOREM.)

CONSISTENT ORDERING

11: page 28

In the successive computation of a set of values $\{u_1,\ldots,u_N\}$ using the GAUSS-SEIDEL or SOR iterative scheme, a scanning order of the points 1,2,...,N is a consistent ordering if it yields the same set of equations (at each iteration) as would be obtained from some matrix which can be partitioned as

$$\begin{bmatrix} D_1 & F_1 & & & \\ E_1 & D_2 & F_2 & & \\ & \cdot & \cdot & \cdot & \\ & & E_{m-2} & D_{m-1} & F_{m-1} \\ & & & E_{m-1} & D_m \end{bmatrix}$$

where the D_i are diagonal matrices and $m \geq 2$.

CONTINUOUS WITH RESPECT TO DATA

W : page 6

A problem is continuous with respect to its data if solutions corresponding to data which differ by small amounts also differ by small amounts.

CONTINUOUSLY DIFFERENTIABLE

A function f is continuously differentiable on the interval I if its derived function f' is continuous on I.

CONVERGENCE IN THE MEAN

W: page 71

The infinite sequence of functions $s_1, s_2, \ldots$ converges in the mean to f (on the interval [a,b]) with respect to the positive weight function ρ if

$$\lim_{N\to\infty} \int_a^b (f-s_N)^2 \rho = 0.$$

Let $\{\phi_n\}$ be an infinite sequence of functions ORTHOGONAL on [a,b] with respect to the weight function ρ. For any function f let $s_N = \Sigma_{n=1}^N c_n \phi_n$ be the sum of the first N terms of the FOURIER SERIES of f. If the limit above is zero for every function f for which $\int_a^b f^2 \rho$ is finite, the set $\{\phi_n\}$ is said to be complete.

CONVERGENCE OF A FINITE-DIFFERENCE REPLACEMENT

8: page 8

A FINITE-DIFFERENCE REPLACEMENT of a differential equation is convergent if, as the MESH spacings tend to zero, the finite-difference solution tends to the true solution of the differential equation either at a fixed point, or for all points along the furthest time level under consideration.

CONVEX

3: page 14

A subset of R^2 (or R^3) is convex if no straight line intersects its boundary in more than two points, except possibly along a straight-line (or plane) section of the boundary.

CURVE

$\underset{\sim}{2}$: page 18

$\underset{\sim}{W}$: page 49

A curve C in R^2 is given by

$$\{(x,y):\ x = \phi(\tau), y = \theta(\tau) \qquad \tau_0 \le \tau \le \tau_1\}$$

where ϕ and θ are continuous. If ϕ and θ are CONTINUOUSLY DIFFERENTIABLE on $[\tau_0,\tau_1]$ with $\theta'^2 + \phi'^2 > 0$ we say that C is a continuously differentiable curve. If these conditions hold except for a finite number of values of τ, C is a piecewise continuously differentiable curve. The curve C is closed if

$$\phi(\tau_1) = \phi(\tau_0) \text{ and } \theta(\tau_1) = \theta(\tau_0).$$

D'ALEMBERT'S SOLUTION

$\underset{\sim}{W}$: pages 9,13

D'Alembert's solution is the general solution of the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

in the form $u(x,t) = p(x+ct) + q(x-ct)$, where p and q are arbitrary functions.

DELTA FUNCTION

$\underset{\sim}{10}$: page 15

The delta function in n dimensions is defined by

$$\delta(\underset{\sim}{r}-\underset{\sim}{\rho}) = 0 \qquad \underset{\sim}{r} \ne \underset{\sim}{\rho}$$

$$\int_D f(\underset{\sim}{r})\delta(\underset{\sim}{r}-\underset{\sim}{\rho})d\underset{\sim}{r} = \begin{cases} f(\underset{\sim}{\rho}) & \underset{\sim}{\rho} \in D \\ 0 & \underset{\sim}{\rho} \notin D \end{cases}$$

for all functions f and all domains $D \subset R^n$.

DIFFUSION EQUATION

$\underset{\sim}{3}$: page 31

The diffusion equation is

$$\frac{\partial u}{\partial t} - k\nabla^2 u = 0.$$

DIRECTIONAL DERIVATIVE

$\underset{\sim}{3}$: page 7

The directional derivative of the SCALAR FIELD ϕ in the direction of the unit vector $\underset{\sim}{e}$ is given by the inner product $\underset{\sim}{e}\cdot\underset{\sim}{\mathrm{grad}}\phi$.

DIRICHLET PROBLEM
3: page 9
9: page 14

A boundary value problem in which a solution, u, of a differential equation in a DOMAIN is specified on the boundary C of the domain is called a *Dirichlet problem*. If, instead of u, the normal derivative $\partial u/\partial n$ is specified on C, the problem is called a *Neumann problem*.

DISCRETIZATION
8: page 17

Discretization is the replacement of a derivative by a finite-difference formula (e.g., the FORWARD-DIFFERENCE FORMULA).

DISPERSION
7: page 13

The physical phenomenon of the velocity of a wave being dependent upon its frequency is called *dispersion*.

DISPLACEMENT VECTOR
11: page 20
S: page 78

The displacement vector $\boldsymbol{\Delta}^{(n)}$ is the correction to $\mathbf{x}^{(n)}$ at the (n+1)th iteration of an ITERATIVE SCHEME, i.e.,

$$\boldsymbol{\Delta}^{(n)} = \mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}.$$

DOMAIN (1)

The *domain* of a given function f is the set of values for which f is defined.

DOMAIN (2)
W: page 49

A connected open subset of R^n is called a *domain*. (A domain in one dimension is just an open interval.)

DOMAIN OF DEPENDENCE
2: page 12

The *domain of dependence* of the point $(\bar{x},\bar{t})$ for a given HYPERBOLIC equation is the set of all points in the solution domain whose DOMAINS OF INFLUENCE include the point $(\bar{x},\bar{t})$.

DOMAIN OF INFLUENCE $\underset{\sim}{2}$: page 12 $\underset{\sim}{W}$: page 40	For a HYPERBOLIC equation, the two CHARACTERISTIC curves through the point $(\bar{x},\bar{t})$ bound a domain which is called the <u>domain of influence</u> of the point $(\bar{x},\bar{t})$. Values of the solution at $\bar{x}$ for a given time $\bar{t}$ can affect the solution only at points within this domain for which $t > \bar{t}$.
EIGENFUNCTION	See EIGENVALUE.
EIGENVALUE $\underset{\sim}{W}$: pages 65,160 $\underset{\sim}{S}$: page 62	Given a linear problem consisting of the equation $Lu + \lambda\rho u = 0$ (where L is a LINEAR OPERATOR) with homogeneous BOUNDARY CONDITIONS, it sometimes turns out that nontrivial solutions are possible only if λ takes on prescribed values called <u>eigenvalues</u>. For a given problem, the solutions associated with each eigenvalue are called <u>eigenfunctions</u> (in the case of a function space) or <u>eigenvectors</u>.
ELLIPTIC	See CLASSIFICATION OF OPERATORS.
EQUATION OF STATE $\underset{\sim}{1}$: page 15	The <u>equation of state</u> of a fluid is a relation between pressure and density. For an ideal gas, we have the adiabatic equation $p\rho^{-\gamma}$ = constant.
EVEN FUNCTION $\underset{\sim}{W}$: page 23	A function f with domain R or [-a,a] or (-a,a) is <u>even</u> if, for each x in the domain, $$f(x) = f(-x).$$

EXPLICIT SCHEME — A FINITE-DIFFERENCE SCHEME is <u>explicit</u> if each equation expresses one unknown PIVOTAL VALUE in terms of known pivotal values.

S : page 11

FINITE-DIFFERENCE REPLACEMENT — A <u>finite-difference replacement</u> of a differential equation is a FINITE-DIFFERENCE SCHEME obtained by DISCRETIZATION of the derivatives in the equation.

S : page 11

FINITE-DIFFERENCE SCHEME (or METHOD) — A <u>finite-difference scheme</u> is a formula relating the PIVOTAL VALUES of a function over a MESH.

S : page 11

FINITE FOURIER TRANSFORM — The <u>finite Fourier transform</u> of $u(r,\theta)$, where $-\pi \le \theta < \pi$, is the sequence $\{a_n(r), b_n(r)\}$ given by

$$a_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r,\theta)\cos n\theta \, d\theta \quad n = 0,1,2,\ldots$$

$$b_n(r) = \frac{1}{\pi}\int_{-\pi}^{\pi} u(r,\theta)\sin n\theta \, d\theta \quad n = 1,2,\ldots$$

W : page 129

FINITE SINE TRANSFORM — The <u>finite sine transform</u> of $u(r,\theta)$, where $0 < \theta < \pi$, is the sequence $\{b_n(r)\}$ given by

$$b_n(r) = \frac{2}{\pi}\int_{0}^{\pi} u(r,\theta)\sin n\theta \, d\theta \quad n = 1,2,\ldots$$

W : page 127

FLUID EQUATIONS — See EQUATION OF STATE, MASS-CONSERVATION EQUATION and MOMENTUM EQUATION.

FLUX

$\underset{\sim}{3}$: page 21

$\underset{\sim}{16}$: page 22

The flux of a fluid is its rate of flow, i.e., the volume crossing a given surface per unit time. If the velocity of the fluid is given by the VECTOR FIELD $\underset{\sim}{v}$, the flux across the surface S is given by

$$\int_S \underset{\sim}{v}\cdot\underset{\sim}{n}\, dS$$

where $\underset{\sim}{n}$ is a unit vector normal to S. In the case of a fluid which flows down a pipe with speed w the flux through a cross-section D of the pipe is

$$\int_D w\, dA.$$

FORWARD-DIFFERENCE OPERATOR

$\underset{\sim}{5}$: page 9

The forward-difference operator Δ_h is defined by

$$\Delta_h u : x \longmapsto u(x+h) - u(x) \qquad x \in R.$$

FOURIER-BESSEL SERIES

$\underset{\sim}{14}$: page 10

A Fourier-Bessel series is the FOURIER SERIES expansion of an arbitrary function with domain (0,1) as $\Sigma_{k=1}^{\infty} c_k J_m(\sqrt{\lambda_k^{(m)}}\,x)$ in terms of BESSEL FUNCTIONS satisfying $J_m(\sqrt{\lambda_k^{(m)}}) = 0$.

FOURIER COEFFICIENT

See FOURIER SERIES.

FOURIER SERIES

$\underset{\sim}{W}$: page 72

The expansion of an arbitrary SQUARE INTEGRABLE function f as a series $\Sigma_{k=1}^{\infty} c_k \phi_k$ in terms of the ORTHOGONAL set of functions $\{\phi_k\}$, with

$$c_k = \frac{f\cdot\phi_k}{\phi_k\cdot\phi_k},$$

is called a Fourier series with Fourier coefficients $\{c_k\}$.

GAMME FUNCTION

14: page 7

The _Gamma function_ is the function

$$\Gamma : a \longmapsto \int_0^\infty e^{-x} x^{\alpha-1} dx \qquad \alpha > 0$$

GAUSS-SEIDEL ITERATION MATRIX

11: page 23

S: page 78

For the ITERATIVE SCHEME

$$\mathbf{x}^{(n+1)} = L\mathbf{x}^{(n+1)} + U\mathbf{x}^{(n)} + \mathbf{b},$$

the matrix $(I-L)^{-1}U$ is called the _Gauss-Seidel iteration matrix_.

GAUSS-SEIDEL METHOD

11: pages 23,24

S: page 78

The _Gauss-Seidel method_ for solving the problem

$$(I-L-U)\mathbf{x} = \mathbf{b},$$

where L and U are respectively lower and upper triangular matrices, is given by the ITERATIVE SCHEME

$$\mathbf{x}^{(n+1)} = L\mathbf{x}^{(n+1)} + U\mathbf{x}^{(n)} + \mathbf{b}.$$

GLOBAL ERROR

5: page 27

8: page 7

The _global error_, at a MESH point, in a FINITE-DIFFERENCE REPLACEMENT of a differential equation is the difference between the computed finite-difference solution and the true solution to the differential equation, at that point.

GREEN'S FUNCTION FOR ORDINARY DIFFERENTIAL EQUATIONS

W : page 122

10 : page 16

Green's function $G(x,\xi)$ for the boundary value problem

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x) = -f(x) \quad x \in (\alpha,\beta)$$
$$u(\alpha) = u(\beta) = 0$$

is given, for each $\xi \in (\alpha,\beta)$, by the continuous solution of

$$\frac{d}{dx}\left[p(x)\frac{dG}{dx}\right] + q(x)G = 0 \qquad x \neq \xi,$$

$$G\Big|_{x=\alpha} = G\Big|_{x=\beta} = 0,$$

$$G\Big|_{x=\xi+0} - G\Big|_{x=\xi-0} = 0,$$

$$\frac{dG}{dx}\Big|_{x=\xi+0} - \frac{dG}{dx}\Big|_{x=\xi-0} = -\frac{1}{p(\xi)}\ .$$

Alternatively, we can write this system as

$$\frac{d}{dx}\left[p(x)\frac{dG}{dx}\right] + q(x)G = -\,\delta(x-\xi),$$
$$G(\alpha,\xi) = G(\beta,\xi) = 0,$$

where $\delta(x-\xi)$ is the DELTA FUNCTION in one dimension. The solution to the boundary value problem above may be expressed as

$$u(x) = \int_\alpha^\beta G(x,\xi)f(\xi)d\xi.$$

GREEN'S FUNCTION FOR PARTIAL DIFFERENTIAL EQUATIONS

W : page 135

10: page 18

For any DOMAIN D with boundary C, Green's function $G(\mathbf{r};\boldsymbol{\rho})$ for the boundary value problem

$$L[u](\mathbf{r}) = -F(\mathbf{r}) \qquad \mathbf{r} \in D$$
$$u(\mathbf{r}) = 0, \qquad \mathbf{r} \in C$$

where L is a linear ELLIPTIC operator, is given, for each $\boldsymbol{\rho} \in D$, by the continuous solution of

$$L[G](\mathbf{r};\boldsymbol{\rho}) = -\,\delta(\mathbf{r}-\boldsymbol{\rho}) \qquad \mathbf{r} \in D$$
$$G(\mathbf{r};\boldsymbol{\rho}) = 0 \qquad \mathbf{r} \in C.$$

The solution to the problem above may be written in the form

$$u(\mathbf{r}) = \int_D G(\mathbf{r};\boldsymbol{\rho})F(\boldsymbol{\rho})\,d\boldsymbol{\rho}.$$

GRID

See MESH.

HARMONIC

6 : page 14

16: page 20

The nth harmonic of a function f with domain $[-L,L]$ is $a_n \sin n\pi x/L + b_n \cos n\pi x/L$, where a_n and b_n are the appropriate FOURIER COEFFICIENTS of f.

HARMONIC FUNCTION

W : page 52

A solution of Laplace's equation

$$\nabla^2 u = 0$$

is called a harmonic function.

HEAT CONTENT

3 : page 31

Let $u(x,y,z,t)$ be the temperature at the point (x,y,z) in the domain D at time $t \geq 0$. The heat content of D is a measure of the total amount of heat in D and is given by

$$c\rho \iiint_D u(x,y,z,t)\; dxdydz$$

where ρ is the density and c the SPECIFIC HEAT of the material.

HEAT EQUATION

W: page 58

The heat equation is another name for the DIFFUSION EQUATION.

HÖLDER CONTINUOUS

W: page 79

The function $f : I \to R$ is Hölder continuous at $x \in I$ if $\exists M > 0$, $\alpha > 0$ such that $\forall y \in I$

$$|f(y) - f(x)| \le M|y-x|^{\alpha}.$$

HOMOGENEOUS EQUATION

W: page 30

If L is any LINEAR operator then

$$Lx = 0$$

is a homogeneous equation.

HOOKE'S LAW

W: page 6

Hooke's Law states that force exerted by an elongated elastic medium is proportional to the extension per unit length; the constant of proportionality is called Young's Modulus.

HYPERBOLIC

See CLASSIFICATION OF OPERATORS.

IDEAL FLUID

16: page 7

An ideal fluid is one whose coefficient of VISCOSITY is zero.

IMPLICIT SCHEME

S: page 18

A FINITE-DIFFERENCE SCHEME is implicit if it must be solved for several unknown values simultaneously.

IMPROPER INTEGRAL

6: page 14

A integral in which the integrand or the domain of integration is unbounded may be evaluated as an improper integral if it may be realized as the limit of a sequence of (proper) integrals. For example, if f(a+0) does not exist we may define

$$\int_a^b f = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f$$

provided this limit exists. Similarly we define

$$\int_a^\infty f = \lim_{b \to \infty} \int_a^b f$$

provided this limit exists

INDUCED INSTABILITY

S : page 6

A method of solution for a problem suffers from induced instability if the method magnifies LOCAL ERRORS so that the computed result differs significantly from the true result.

INFLUENCE FUNCTION

W : page 119

The influence function $R(x,\xi)$ for the initial value problem

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u(x) = f(x) \qquad x > \alpha,$$

$$u(\alpha) = u'(\alpha) = 0,$$

is given, for each $\xi > \alpha$, by the solution of the initial value problem

$$\frac{d}{dx}\left[p(x)\frac{dR}{dx}\right] + q(x)R = 0 \qquad x > \xi,$$

$$R\Big|_{x=\xi} = 0,$$

$$\frac{dR}{dx}\Big|_{x=\xi} = \frac{1}{p(\xi)},$$

or the equivalent problem

$$\frac{d}{dx}\left[p(x)\frac{dR}{dx}\right] + q(x)R = \delta(x-\xi),$$

$$R\Big|_{x=\xi} = 0.$$

The solution of the original problem is given by

$$u(x) = \int_\alpha^x R(x,\xi)f(\xi)d\xi.$$

INHERENT INSTABILITY

5: page 5

A mathematical problem suffers from inherent instability if it is not CONTINUOUS WITH RESPECT TO ITS DATA or is not WELL CONDITIONED.

INITIAL-BOUNDARY VALUE PROBLEM

See CAUCHY PROBLEM.

INITIAL CONDITIONS

1: page 9

Initial conditions for a partial differential equation are conditions given at spatial points for time $t = t_0$. For example, a stretched string may have its motion started with a given shape, $u(x,0) = f(x)$, and a given velocity distribution $\frac{\partial u}{\partial t}(x,0) = g(x)$. For ordinary differential equations, initial conditions are conditions specified at one point of the solution domain.

INITIAL VALUE PROBLEM

See INFLUENCE FUNCTION.

ITERATIVE SCHEME

11: page 15

An iterative scheme for solving a problem is one in which the solution is obtained by a process of successive approximation.

JACOBIAN

15: page 15

The Jacobian of the N functions $f_1, f_2, \ldots, f_N$ each with domain variable $(x_1, x_2, \ldots, x_N) \in R^N$ and codomain R is the determinant of the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{bmatrix}$$

JACOBI METHOD

11: page 24

S: page 78

The Jacobi method is the ITERATIVE SCHEME

$$\underset{\sim}{x}^{(n+)} = L\underset{\sim}{x}^{(n)} + U\underset{\sim}{x}^{(n)} + \underset{\sim}{b}$$

for the solution of the equation

$$(I-L-U)\underset{\sim}{x} = \underset{\sim}{b},$$

where L and U are respectively lower and upper triangular matrices.

JUMP DISCONTINUITY

6: page 14

The function f has a jump discontinuity at x if the one-sided limits $f(x+0)$ and $f(x-0)$ exist but are not both equal to $f(x)$.

KINETIC ENERGY

2: page 14

The kinetic energy of a particle is the energy it possesses by virtue of its motion, and is defined to be

$$\tfrac{1}{2}(\text{mass}) \times (\text{velocity})^2.$$

KRONECKER DELTA

The Kronecker delta is given by

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad i,j \in Z.$$

LAPLACE'S EQUATION

W: pages 43,49,52

Laplace's equation is $\nabla^2 u = 0$.

LAW OF CONSERVATION OF ENERGY

2: page 13

The Law of Conservation of Energy states that the change in the total energy (= KINETIC ENERGY + POTENTIAL ENERGY) of a system is equal to the work done on the system by all the forces. In particular the energy of an isolated system remains constant.

LAW OF CONSERVATION OF MASS
1: page 12

The Law of Conservation of Mass states that the total mass of a quantity of material remains constant in time, however it becomes distributed in space.

LINE INTEGRAL
3: page 12

The line integral $\int_C f\,ds$ represents $\lim_{\Delta s \to 0} \Sigma f_P \,\Delta s$ over the CURVE C, where f_P is the value of f at a point P in the small element of C with length Δs.

LINEAR (OPERATOR)
W: page 29

L is a linear transformation (operator) if it has the property

$$L[\alpha \underset{\sim}{u} + \beta \underset{\sim}{v}] = \alpha L[\underset{\sim}{u}] + \beta L[\underset{\sim}{v}],$$

where α and β are any constants and $\underset{\sim}{u}, \underset{\sim}{v}$ are any vectors (functions).

LINEAR PARTIAL DIFFERENTIAL EQUATION
W: page 30

$$L[u] = F$$

where L is a LINEAR OPERATOR involving partial differentiation and F is a given function.

LOCAL ERROR
5: page 27

The local error is the error made at a point when a FINITE-DIFFERENCE SCHEME is used once only. In the complete step-by-step process, the finite-difference scheme is used many times so that local errors accumulate to produce GLOBAL ERRORS.

LOCAL TRUNCATION ERROR
5: page 14

The local truncation error (at a point) in a FINITE-DIFFERENCE REPLACEMENT of a differential equation is the LOCAL ERROR introduced by DISCRETIZATION.

MASS-CONSERVATION EQUATION

$\underset{\sim}{1}$: page 13

$\underset{\sim}{14}$: page 26

The Mass-Conservation equation expresses mathematically the physical LAW OF CONSERVATION OF MASS as applied to problems in fluid flow. It is given by

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underset{\sim}{u} = 0,$$

where ρ and $\underset{\sim}{u}$ denote the density and velocity respectively, and t is the time coordinate. In one spatial dimension with coordinate x the equation reduces to

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x} = 0.$$

MEAN SQUARE DEVIATION

$\underset{\sim}{W}$: page 71

The mean square deviation of g from f, with respect to the weight function ρ on [a,b], is

$$\int_a^b (f-g)^2 \rho.$$

MESH

$\underset{\sim}{5}$: page 7

The set of points $\{(x_i, t_j)\} = \{(x_0 + ih, t_0 + jk)\}$ where h and k are constants is called a mesh or grid. In a given mesh each point (x_i, t_j) is called a mesh point, and h and k are called the mesh spacings or mesh lengths in the x- and t-directions respectively.

MESH RATIO

$\underset{\sim}{5}$: pages 11,25

A FINITE-DIFFERENCE REPLACEMENT of a HYPERBOLIC or PARABOLIC differential equation depends on the MESH spacings h and k (in the x- and t-directions respectively) according to the value of the mesh ratio k^2/h^2 (for hyperbolic equations) or k/h^2 (for parabolic equations). For typical examples, see Section 3.

MOLECULAR DIAGRAM $\underset{\sim}{S}$: page 12 $\underset{\sim}{5}$: page 11	A molecular diagram is a diagrammatic representation of a FINITE-DIFFERENCE SCHEME which indicates the coefficients of the PIVOTAL VALUES related by the scheme.
MOMENT $\underset{\sim}{16}$: page 13	The moment about the point P of a force, acting through the point Q and represented by the geometric vector $\underset{\sim}{F}$, is $\underset{\sim}{r} \times \underset{\sim}{F}$ where $\underset{\sim}{r}$ is the geometric vector $\underset{\rightarrow}{PQ}$.
MOMENTUM EQUATION $\underset{\sim}{1}$: page 15 $\underset{\sim}{14}$: page 27	The Momentum equation expresses mathematically NEWTON'S SECOND LAW OF MOTION as applied to fluid flow. It is given by $$\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u}\cdot\underset{\sim}{\text{grad}})\underset{\sim}{u} + \frac{1}{\rho}\,\underset{\sim}{\text{grad}}\ p = \underset{\sim}{F},$$ where $\rho, \underset{\sim}{u}, p$ and $\underset{\sim}{F}$ denote the density, velocity, pressure and BODY FORCE per unit mass respectively, and t is the time coordinate. In one spatial dimension with coordinate x the equation reduces to $$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = F.$$
NEUMANN PROBLEM	See DIRICHLET PROBLEM.
NEWTON	A newton is the SI unit of force.
NEWTONIAN FLUID $\underset{\sim}{16}$: page 7	A Newtonian fluid is one whose coefficient of VISCOSITY does not depend on the velocity gradient.

NEWTON'S SECOND LAW OF MOTION
$\underset{\sim}{1}$: page 7

<u>Newton's Second Law of Motion</u> states that

force = rate of change of linear momentum

= mass × acceleration (for a body of constant mass)

NUMERICAL CHARACTERISTICS
$\underset{\sim}{5}$: page 13

The <u>numerical characteristics</u> of a FINITE-DIFFERENCE SCHEME are the lines which bound the NUMERICAL DOMAIN OF INFLUENCE.

NUMERICAL DOMAIN OF DEPENDENCE
$\underset{\sim}{8}$: page 12

The <u>numerical domain of dependence</u> of a MESH point $(\bar{x},\bar{t})$ for a given FINITE-DIFFERENCE SCHEME is the set of mesh points whose NUMERICAL DOMAINS OF INFLUENCE include $(\bar{x},\bar{t})$.

NUMERICAL DOMAIN OF INFLUENCE
$\underset{\sim}{5}$: page 12

The <u>numerical domain of influence</u> of a MESH point $(\bar{x},\bar{t})$ for a given FINITE-DIFFERENCE SCHEME consists of all those mesh points at which the PIVOTAL VALUES of the solution depend on the value of the solution at $(\bar{x},\bar{t})$.

ODD FUNCTION
$\underset{\sim}{W}$: page 12

A function f with domain R or $[-a,a]$ or $(-a,a)$ is <u>odd</u> if, for each x in the domain,

$$f(-x) = -f(x).$$

ORTHOGONAL
$\underset{\sim}{W}$: page 71

Two elements, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ of an inner product space are <u>orthogonal</u> if $\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$. For example, two real functions ϕ and ψ are <u>orthogonal with respect to the weight function ρ</u> on (α,β) if

$$\int_\alpha^\beta \phi\psi\rho = 0.$$

ORTHOGONAL BASIS

An <u>orthogonal basis</u> is a COMPLETE set of ORTHOGONAL functions.

ORTHONORMAL FUNCTIONS 6 : page 12	A set of functions $\{\phi_n\}$ which satisfies $$\int_\alpha^\beta \phi_m(x)\phi_n(x)\rho(x)dx = \delta_{mn},$$ where δ_{mn} is the KRONECKER DELTA, is said to be orthonormal with respect to the weight function ρ on the interval (α,β).
OVER-RELAXATION FACTOR	See SUCCESSIVE OVER-RELAXATION METHOD.
PARABOLIC	See CLASSIFICATION OF EQUATIONS.
PERIODIC FUNCTION	A function f is periodic (with period a) if, for all $x \in R$, $f(x) = f(x+a)$.
PIECEWISE CONTINUOUS 6 : page 15	A function is piecewise continuous on $[a,b]$ if it is continuous at each point of $[a,b]$ except for a finite number of JUMP DISCONTINUITIES.
PIECEWISE CONTINUOUSLY DIFFERENTIABLE CURVE	See CURVE.
PIVOTAL VALUE 5 : page 7	The values of a function u at MESH points are called pivotal values of u.
POINTWISE CONVERGENCE (OF A SERIES) W : page 70	Given an infinite sequence of functions $\phi_1,\phi_2,\ldots$ with domain I, we let $s_N(x) = \Sigma_{n=1}^N c_n\phi_n(x)$. If for each $x \in I$ $\lim_{N\to\infty} s_N(x) = f(x)$, we say that the series $\Sigma_{n=1}^\infty c_n\phi_n(x)$ converges to f(x) pointwise in I.

POISE

16: page 6

A poise is the cgs unit of VISCOSITY.

POISSON'S EQUATION

W : pages 49,50,129

3 : page 5

Poisson's equation is

$$\nabla^2 u = -F.$$

POTENTIAL ENERGY

2 : page 14

The potential energy of a system is the energy stored within the system.

PROPERLY POSED PROBLEM

W : page 6

3 : page 10

A system consisting of a partial differential equation together with (INITIAL and) BOUNDARY CONDITIONS is properly posed if its solution (a) exists, (b) is unique and (c) is CONTINUOUS WITH RESPECT TO THE DATA.

PROPERTY A

11: page 29

A square matrix is said to have Property A if, by transposing pairs of rows and pairs of corresponding columns, it can be transformed to the form of the matrix

$$\begin{bmatrix} D_1 & F \\ E & D_2 \end{bmatrix}$$

where D_1 and D_2 are diagonal matrices.

RAYLEIGH QUOTIENT

W : pages 163,165

Consider the differential equation

$$(pu')' - qu + \lambda\rho u = 0 \quad \text{in } (\alpha,\beta),$$

where q and ρ are continuous and p is CONTINUOUSLY DIFFERENTIABLE in (α,β), along with $u(\alpha) = u(\beta) = 0$. Let ϕ vanish at α and β and be twice continuously differentiable in (α,β). The Rayleigh quotient of ϕ is given by the ratio

$$\frac{\int_\alpha^\beta (p\phi'^2+q\phi^2)}{\int_\alpha^\beta \rho\phi^2}$$

Characterizations of the EIGENVALUES of the problem above as minima of the Rayleigh quotient are called minimum principles for the eigenvalues.

RECURRENCE RELATION

A recurrence relation is an equation relating successive members of a sequence.

ROUNDING ERROR

A rounding error is introduced into the solution of a problem by working to a definite number of significant figures.

RESIDUAL VECTOR

11: page 16

S: pages 31,79

Given a system of linear equations

$$A\underset{\sim}{x} = \underset{\sim}{b},$$

then, if $\bar{\underset{\sim}{x}}$ is an approximation to $\underset{\sim}{x}$, the vector

$$\underset{\sim}{r} = \underset{\sim}{b} - A\bar{\underset{\sim}{x}}$$

is called the residual vector.

SCALAR FIELD

3: page 6

Let D denote a subset of R^3; then a function

$$\phi : D \to R \qquad \text{or} \qquad \phi : D \times R_0^+ \to R$$
$$\phi : (x,y,z) \longmapsto u \qquad\qquad \phi : (x,y,z,t) \longmapsto u$$

is called a scalar field. (In the second case we say that ϕ is time-dependent.)

SELF-ADJOINT FORM

W: page 117

An ordinary differential equation written in the form

$$(pu')' + qu = f$$

on the interval (a,b) is said to be in self-adjoint form. We require that p be CONTINUOUSLY DIFFERENTIABLE and positive and that q and f be continuous on [a,b] .

SINGULAR POINT

10: page 7

A singular point of the ordinary differential equation

$$a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u = 0$$

is a point x where either $a(x) = 0$ or $b(x)$ or $c(x)$ is infinite.

SOR

See SUCCESSIVE OVER-RELAXATION METHOD.

SPARSE

11 : page 15

A matrix is sparse if it has a large proportion of zero elements.

SPECIFIC HEAT

3 : page 31

The specific heat of a material is the quantity of heat required to raise the temperature of unit mass of the material by one unit.

SPECTRAL RADIUS

8 : page 18

S : page 79

The spectral radius of the matrix M is

$$\rho(M) = \max_i |\lambda_i|$$

where λ_i $(i=1,2,\ldots,n)$ are the EIGENVALUES of M.

SQUARE INTEGRABLE

6 : page 13

The function f is square integrable on (α,β) with respect to the weight function ρ if the integral

$$\int_\alpha^\beta f^2\rho$$

exists.

STABLE

8 : page 17

A FINITE-DIFFERENCE SCHEME is stable if LOCAL ERRORS do not accumulate exponentially as the step-by-step solution progresses, i.e., if it does not suffer from INDUCED INSTABILITY.

STANDARD FORM

W: pages 42,43

The standard form (or canonical form) for a linear partial differential operator which is

(a) HYPERBOLIC is $\frac{\partial^2 u}{\partial\xi\partial\eta} + F(\frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}, u, \xi, \eta) = 0$;

(b) PARABOLIC is $\frac{\partial^2 u}{\partial\eta^2} + F(\frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}, u, \xi, \eta) = 0$;

(c) ELLIPTIC is $\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2} + F(\frac{\partial u}{\partial\xi}, \frac{\partial u}{\partial\eta}, u, \xi, \eta) = 0$.

STANDING WAVE

1 : page 35

2 : page 9

A standing wave is a wave which does not move laterally, but merely changes its amplitude with time. A standing wave is easily recognized by observing points of zero displacement which remain stationary for all time, e.g., $A\sin\omega t\cos kx$.

SUBSIDIARY CONDITIONS

Subsidiary conditions are INITIAL and/or BOUNDARY CONDITIONS.

SUCCESSIVE OVER-RELAXATION METHOD (SOR)

11: page 26

The successive over-relaxation method for solving

$$(I-L-U)\underset{\sim}{x} = \underset{\sim}{b},$$

where L and U are respectively lower and upper triangular matrices, is given by the ITERATIVE SCHEME

$$(I-\omega L)\underset{\sim}{x}^{(n+1)} = \{(1-\omega)I + \omega U\}\underset{\sim}{x}^{(n)} + \omega\underset{\sim}{b}.$$

The parameter ω is called the over-relaxation factor, and determines the ASYMPTOTIC RATE OF CONVERGENCE of the method. The optimum over-relaxation factor is that value which maximizes the rate of convergence of the method.

SURFACE INTEGRAL

3: pages 11,27

The surface integral $\int_S f\,dA$ of the function f over the surface S represents $\lim_{\Delta A \to 0} \Sigma f_P \Delta A$, where f_P is the value of f at a point P in the small element of S with area ΔA.

TAYLOR APPROXIMATION

1: page 13

2: page 21

14: page 26

A Taylor approximation of a function is obtained by neglecting the remainder term in TAYLOR'S THEOREM (see Section 3). The first-order Taylor approximations for functions of one, two and three variables are respectively:

$$u(x+h) \simeq u(x) + hu'(x);$$

$$u(x+h,t+h) \simeq \left[u + h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial t}\right]_{(x,t)};$$

$$u(x+\Delta x, y+\Delta y, z+\Delta z) \simeq [u + \Delta\underset{\sim}{r}\cdot\underset{\sim}{\mathrm{grad}}\ u]_{(x,y,z)}$$

where $\Delta\underset{\sim}{r} = (\Delta x, \Delta y, \Delta z)$.

THERMAL CONDUCTIVITY

W: page 58

The thermal conductivity of a material is the rate at which heat flows across a temperature gradient of unit magnitude.

THREE-LEVEL SCHEME

15: page 6

A three-level FINITE-DIFFERENCE SCHEME involves terms along the (j-1)th, jth and (j+1)th time levels. If it is linear, it may be written in matrix form as

$$A\underset{\sim}{u}_{j+1} = B\underset{\sim}{u}_j + C\underset{\sim}{u}_{j-1}.$$

TRAVELLING WAVE

1: page 20

A travelling wave is a wave which moves in a given direction without changing its shape. For example $f(x+ct)$ represents a wave travelling in the negative x-direction. The substitution $X = x+ct$ gives $f(X)$ which may represent a wave of fixed shape. Since the origin of the X-coordinate is at $x = -ct$, the wave travels with velocity c in the negative x-direction. Similarly, $f(x-ct)$ represents a travelling wave of velocity c in the positive x-direction.

TRIDIAGONAL MATRIX

5: page 18

A tridiagonal matrix is one whose nonzero elements appear on and adjacent to its main diagonal, i.e., a BAND MATRIX of BAND WIDTH 3.

UNIFORMLY BOUNDED

6: page 21

The infinite sequence of functions $\{b_n\}$ is uniformly bounded on the interval I if $\exists c \in R$, independent of $n \in Z^+$ and $x \in I$, such that $n \in Z^+$ and $x \in I$, $|b_n(x)| < c$.

UNIFORM CONVERGENCE

2: page 10

6: page 9

W: page 70

The infinite sequence of functions $\{f_n\}$ converges uniformly to f on the interval I if it is convergent to f in the uniform norm, i.e., if for each $\varepsilon > 0 \ \exists N_\varepsilon \in Z^+$ such that

$$\max_{x \in I} |f(x) - f_N(x)| < \varepsilon$$

whenever $N \geq N_\varepsilon$. An equivalent formulation is: for each $\varepsilon > 0, \exists N_\varepsilon \in Z^+$ independent of x such that $|f(x) - f_N(x)| < \varepsilon$ for all $x \in I$ whenever $N \geq N_\varepsilon$.

The infinite series of functions $\Sigma_{n=1}^{\infty} c_n \phi_n$ converges uniformly on the interval I if the sequence $\{s_N\}$ of partial sums

$$s_N = \sum_{n=1}^{N} c_n \phi_n$$

is uniformly convergent.

VECTOR FIELD

3: page 6

A vector field in two (or three) dimensions is a function with domain D in R^2 (or R^3) and codomain the space G^2 (or G^3) of geometric vectors. A time-dependent vector field is defined similarly with domain $D \times R_0^+$.

VELOCITY POTENTIAL

14: page 15

If the velocity $\underset{\sim}{v}$ of a fluid is related to a SCALAR FIELD Φ by $\underset{\sim}{v} = -\text{grad}\Phi$, Φ is called the velocity potential of the flow, which is then said to be irrotational.

VISCOSITY

3: page 20

16: page 6

If a relative motion occurs in a fluid, a measurable resistance is experienced, and the fluid is said to exhibit viscosity or internal friction. For a fluid moving in parallel layers, the frictional force F on a plane surface of area A parallel to the fluid flow is given to a good approximation by

$$\frac{F}{A} = \mu \frac{\partial v}{\partial z}$$

where $\frac{\partial v}{\partial z}$ is the velocity gradient perpendicular to the direction of motion and μ is called the coefficient of viscosity. If ρ is the density of the fluid, then its kinematic viscosity is μ/ρ.

WAVE EQUATION $\underset{\sim}{W}$: pages 9,36,48,66,152 $\underset{\sim}{1}$: page 8	The wave equation is $$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u.$$
WAVE LENGTH	See ANGULAR FREQUENCY.
WAVE NUMBER	See ANGULAR FREQUENCY.
WELL CONDITIONED $\underset{\sim}{11}$: page 15	The linear equation $$A\underset{\sim}{u} = \underset{\sim}{b}$$ is well conditioned if small changes in the elements of A and $\underset{\sim}{b}$ do not cause large changes in the solution vector $\underset{\sim}{u}$.
YOUNG'S MODULUS	See HOOKE'S LAW.

2 VECTORS

A **vector space** V over a *field F of* **scalars** is a set of elements with an internal binary operation $+ : V \times V \longrightarrow V$ and an external binary operation of multiplication $F \times V \longrightarrow V$ satisfying the following axioms for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ and $m, n \in F$:

1 $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

2 $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

3 $\exists \mathbf{0} \in V$ such that $\mathbf{0} + \mathbf{a} = \mathbf{a} \quad \forall \mathbf{a} \in V$

4 $\exists -\mathbf{a} \in V$ such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

5 $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

6 $(m + n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$

7 $(mn)\mathbf{a} = m(n\mathbf{a})$

8 $1\mathbf{a} = \mathbf{a}$

When $F = R$ (the reals) we talk about a *real vector space*: when $F = C$ (the complex numbers) we have a *complex vector space*.

An important notion in a vector space is that of linear independence. A set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of vectors in V is **linearly independent** if

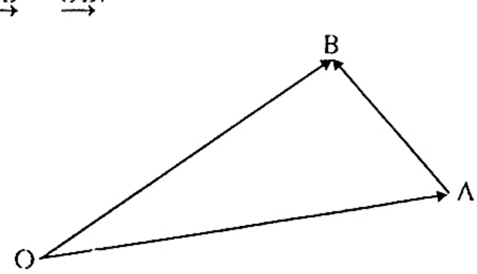
$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \qquad (\alpha_i \in F)$$

holds only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Suppose now that $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a *maximal* linearly independent set in V, i.e. if another vector in V is included in the set we obtain a set which is linearly dependent. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **basis** for V, and every other basis for V has precisely m vectors. We say that V has **dimension** m. It can also be shown that an arbitrary element $\mathbf{x} \in V$ can be expressed uniquely in the form

$$\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \qquad (x_i \in F).$$

Geometric vectors are equivalence classes of parallel arrows of equal length. If $\overrightarrow{AB}$ is an arrow, we write $\underrightarrow{AB}$ for the geometric vector to which it belongs. The sum of two geometric vectors is obtained by choosing suitable arrows which can be added according to the *triangle law of addition*:

$$\underrightarrow{OA} + \underrightarrow{AB} = \underrightarrow{OB}.$$



If λ is a real number then we define the product $\lambda\underrightarrow{AB}$ to be the geometric vector whose arrows are parallel to those of $\underrightarrow{AB}$ but with lengths multiplied by λ.

With respect to these definitions of addition and scalar multiplication the set G^3 of geometric vectors in space forms a real vector space. We often consider the set G^2 of geometric vectors in the plane, which is a *subspace* of this vector space. The null vector $\mathbf{0}$ is the geometric vector whose arrows are all of zero length.

Let $Oxyz$ be a frame of rectangular Cartesian coordinate axes and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be geometric vectors whose arrows are of unit length and parallel to the coordinate axes Ox, Oy, Oz respectively. Then every geometric vector in the plane can be written uniquely in the form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j},$$

and every geometric vector in space can be written uniquely in the form

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Thus $\{\mathbf{i}, \mathbf{j}\}$ forms a basis for the space G^2 of geometric vectors in the plane, and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ forms a basis for G^3; a_1, a_2, a_3 are, respectively, the x-, y-, z-**coordinates** of $\mathbf{a}$.

If I is an interval on the real line or a domain in R^2 or R^3, then the set of all functions $I \to R$ is a real vector space with addition and scalar multiplication of functions given by

$$(f+g) : x \longmapsto f(x) + g(x) \qquad x \in I,$$

$$\forall \lambda \in R \qquad (\lambda f) : x \longmapsto \lambda[f(x)] \qquad x \in I.$$

Replacing R by C in the above definition yields a complex vector space of functions.

The space of all VECTOR FIELDS with domain D forms a real vector space with addition and scalar multiplication of fields given by

$$(\underset{\sim}{u}+\underset{\sim}{v}) : \underset{\sim}{r} \longmapsto \underset{\sim}{u}(\underset{\sim}{r}) + \underset{\sim}{v}(\underset{\sim}{r}) \qquad \underset{\sim}{r} \in D,$$

$$\forall \lambda \in R \qquad \lambda \underset{\sim}{v} : \underset{\sim}{r} \longmapsto \lambda[\underset{\sim}{v}(\underset{\sim}{r})] \qquad \underset{\sim}{r} \in D.$$

An important property of a geometric vector is its *length* or *magnitude*. To define this more generally, we introduce the inner product.

A mapping $\cdot : V \times V \longrightarrow R$, where V is a real vector space, is called a (real) **inner product** if for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ and $\lambda, \mu \in R$:

1 $\mathbf{a} \cdot \mathbf{a} = 0 \Leftrightarrow \mathbf{a} = 0$;

2 $\mathbf{a} \cdot \mathbf{a} \geq 0$;

3 $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;

4 $\mathbf{a} \cdot (\lambda \mathbf{b} + \mu \mathbf{c}) = \lambda \mathbf{a} \cdot \mathbf{b} + \mu \mathbf{a} \cdot \mathbf{c}$.

We may specify an inner product on G^2 or G^3 by

$$\underset{\longrightarrow}{OA} \cdot \underset{\longrightarrow}{OB} = OA \times OB \times \cos\theta,$$

where OA, OB denotes the length of the arrow $\overrightarrow{OA}$, $\overrightarrow{OB}$ respectively and θ is the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The geometric vectors **i**, **j**, **k** are *orthogonal*, i.e.

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0.$$

(In fact, any two geometric vectors which are represented by perpendicular arrows are orthogonal.) If $\underset{\longrightarrow}{OA} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$ and $\underset{\longrightarrow}{OB} = b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k}$, then

$$\underset{\longrightarrow}{OA} \cdot \underset{\longrightarrow}{OB} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

in terms of coordinates.

Given the vector space of SQUARE INTEGRABLE functions with domain (α, β) and codomain R, the map

$$(f, g) \longmapsto \int_\alpha^\beta fg$$

specifies an inner product.

A mapping $\cdot : V \times V \to C$, where V is a complex vector space, is called a complex inner product if for all $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c} \in V$ and $\lambda,\mu \in C$:

1 $\underset{\sim}{a}\cdot\underset{\sim}{a} = 0 \Longleftrightarrow \underset{\sim}{a} = \underset{\sim}{0}$;

2 $\underset{\sim}{a}\cdot\underset{\sim}{a} \geq 0$;

3 $\underset{\sim}{a}\cdot\underset{\sim}{b} = \overline{\underset{\sim}{b}\cdot\underset{\sim}{a}}$;

4 $\underset{\sim}{a}\cdot(\lambda\underset{\sim}{b}+\mu\underset{\sim}{c}) = \lambda\underset{\sim}{a}\cdot\underset{\sim}{b} + \mu\underset{\sim}{a}\cdot\underset{\sim}{c}$.

Schwarz's Inequality states that, given a real inner product space V,

$$\underset{\sim}{x}\cdot\underset{\sim}{y} \leq (\underset{\sim}{x}\cdot\underset{\sim}{x})^{\frac{1}{2}}(\underset{\sim}{y}\cdot\underset{\sim}{y})^{\frac{1}{2}}$$

for all $\underset{\sim}{x},\underset{\sim}{y} \in V$. For the space of square integrable functions with the inner product given previously, this becomes

$$\int_{\alpha}^{\beta} fg \leq \{\int_{\alpha}^{\beta} f^2\}^{\frac{1}{2}}\{\int_{\alpha}^{\beta} g^2\}^{\frac{1}{2}}$$

A mapping $|| \quad || : V \to R$, where V is a real vector space, is called a norm if for all $\underset{\sim}{a},\underset{\sim}{b} \in V$ and $\lambda \in R$:

1 $||\underset{\sim}{a}|| = 0 \Longleftrightarrow \underset{\sim}{a} = \underset{\sim}{0}$;

2 $||a|| \geq 0$;

3 $||\lambda\underset{\sim}{a}|| = |\lambda| \, ||\underset{\sim}{a}||$

4 $||\underset{\sim}{a}+\underset{\sim}{b}|| \leq ||\underset{\sim}{a}|| + ||\underset{\sim}{b}||$ (Triangle Inequality).

In an inner product space the specification

$$||\underset{\sim}{a}|| = (\underset{\sim}{a}\cdot\underset{\sim}{a})^{\frac{1}{2}}$$

yields a norm satisfying the axioms above. The norm of a geometric vector, given in this way, is just the length of an arrow representing it; by convention, if $\underset{\sim}{a}$ represents a geometric vector (or a vector field) then its length is denoted by $|\underset{\sim}{a}|$ or just a.

In G^3 (but not in G^2) we define the **vector product** $\times : G^3 \times G^3 \to G^3$ by

$$\underset{\rightarrow}{OA} \times \underset{\rightarrow}{OB} = |\overrightarrow{OA}|\,|\overrightarrow{OB}| \sin\theta\, \mathbf{n},$$

where θ is the angle $(0 \le \theta \le \pi)$ between $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and $\mathbf{n}$ is the *unit vector* (i.e. vector whose norm is 1) orthogonal to $\underset{\rightarrow}{OA}$ and $\underset{\rightarrow}{OB}$, such that $\underset{\rightarrow}{OA}$, $\underset{\rightarrow}{OB}$ and $\mathbf{n}$ form a right-handed triad.

The vector product has the following properties:

1 $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$;

2 $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$;

3 $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$;

4 $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$;

5 $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, in general.

6 $$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

7 $$\mathbf{a} \times (\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$$

A point in space with Cartesian coordinates (x,y,z) may also be specified by its <u>spherical polar</u> coordinates (r,θ,ϕ) or by its <u>cylindrical polar</u> coordinates (ρ,ϕ,z). These are related to each other as follows.

$$r^2 = x^2 + y^2 + z^2 \qquad\qquad \rho^2 = x^2 + y^2$$

$$x = r \sin\theta \cos\phi = \rho \cos\phi$$

$$y = r \sin\theta \sin\phi = \rho \sin\phi$$

$$z = r \cos\theta$$

In evaluating a VOLUME INTEGRAL the volume element dV is replaced by

$dx\,dy\,dz$	in Cartesian coordinates
$r^2 \sin\theta\, dr\, d\theta\, d\phi$	in spherical polar coordinates
$\rho\, d\rho\, d\phi\, dz$	in cylindrical polar coordinates

The slope of a SCALAR FIELD Ψ is given by the VECTOR FIELD

$$\underset{\sim}{\mathrm{grad}}\ \Psi = \frac{\partial\Psi}{\partial x}\underset{\sim}{i} + \frac{\partial\Psi}{\partial y}\underset{\sim}{j} + \frac{\partial\Psi}{\partial z}\underset{\sim}{k}$$

$$= \frac{\partial\Psi}{\partial r}\underset{\sim}{e}_r + \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\underset{\sim}{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial\phi}\underset{\sim}{e}_\phi$$

$$= \frac{\partial\Psi}{\partial\rho}\underset{\sim}{e}_\rho + \frac{1}{\rho}\frac{\partial\Psi}{\partial\phi}\underset{\sim}{e}_\phi + \frac{\partial\Psi}{\partial z}\underset{\sim}{k},$$

where $\underset{\sim}{e}_\alpha$ denotes the vector field of unit norm in the direction of increasing α.

If $\underset{\sim}{v}$ is a VECTOR FIELD given by

$$\underset{\sim}{v} = v_x\underset{\sim}{i} + v_y\underset{\sim}{j} + v_z\underset{\sim}{k}$$

$$= v_r\underset{\sim}{e}_r + v_\theta\underset{\sim}{e}_\theta + v_\phi\underset{\sim}{e}_\phi$$

$$= v_\rho\underset{\sim}{e}_\rho + v_\phi\underset{\sim}{e}_\phi + v_z\underset{\sim}{k},$$

we may define the SCALAR FIELD

$$\mathrm{div}\ \underset{\sim}{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}\ .$$

The LAPLACIAN operator is given by

$$\nabla^2\Psi = \mathrm{div}\ \underset{\sim}{\mathrm{grad}}\ \Psi$$

$$= \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\phi^2} + \frac{\partial^2\Psi}{\partial z^2}\ .$$

The following rules are useful:

$$\underset{\sim}{\mathrm{grad}}(\Psi\Phi) = \Psi\ \underset{\sim}{\mathrm{grad}}\ \Phi + \Phi\ \underset{\sim}{\mathrm{grad}}\ \Psi$$

$$\mathrm{div}(\Psi \underset{\sim}{v}) = \Psi\ \mathrm{div}\ \underset{\sim}{v} + \underset{\sim}{v}\cdot\underset{\sim}{\mathrm{grad}}\ \Psi$$

$$\mathrm{div}(\Psi\ \underset{\sim}{\mathrm{grad}}\ \Phi) = \Psi\ \nabla^2\Phi + \underset{\sim}{\mathrm{grad}}\ \Psi\cdot\underset{\sim}{\mathrm{grad}}\ \Phi$$

Divergence Theorem (3: pages 14,28)

Let D be a CONVEX DOMAIN in R^2 (or R^3) bounded by the closed CURVE (or closed surface) C. If the coordinates of the VECTOR FIELD $\underset{\sim}{v}$ are continuous on D∪C and their first partial derivatives are continuous in D then

$$\int_D \mathrm{div}\ \underset{\sim}{v}\ dA = \oint \underset{\sim}{v}\cdot\underset{\sim}{n}\ ds \qquad (D \subset R^2),$$

$$\int_D \mathrm{div}\ \underset{\sim}{v}\ dV = \int_C \underset{\sim}{v}\cdot\underset{\sim}{n}\ dS \qquad (D \subset R^3),$$

where $\underset{\sim}{n}$ is the unit outward normal to C.

Green's Theorem (W: page 53)

Let D be a CONVEX DOMAIN in R^2 (or R^3) bounded by the closed CURVE (or closed surface) C. If the SCALAR FIELD u has continuous first partial derivatives on D∪C and continuous second partial derivatives in D then

$$\int_D u\ \nabla^2 u\ dA = \oint_C u\ \frac{\partial u}{\partial n}\ ds - \int_D |\underset{\sim}{\mathrm{grad}}\ u|^2 dA \qquad (D \subset R^2),$$

$$\int_D u\ \nabla^2 u\ dV = \int_C u\ \frac{\partial u}{\partial n}\ dS - \int_D |\mathrm{grad}\ u|^2 dV \qquad (D \subset R^3),$$

where $\partial u/\partial n$ is the outward normal derivative of u on C.

3 USEFUL FORMULAS AND THEOREMS

Trigonometric Functions

$$\sin(\frac{\pi}{2} - \theta) = \cos\theta$$

$$\cos(\frac{\pi}{2} - \theta) = \sin\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$\sin\alpha - \sin\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)$$

$$\cos\alpha + \cos\beta = 2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

$$\cos\alpha - \cos\beta = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta)$$

$$\cot\theta + \tan\theta = 2\operatorname{cosec}2\theta$$

$$\cot\theta - \tan\theta = 2\cot 2\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \text{(Euler's formula)}$$

Hyperbolic Functions

$$\sinh x = \tfrac{1}{2}(e^{x} - e^{-x}) = -i \sin ix$$

$$\cosh x = \tfrac{1}{2}(e^{x} + e^{-x}) = \cos ix$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

Trigonometric Fourier Series

The trigonometric Fourier series of the function f with domain $[-L,L]$ is given by

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos \frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin \frac{n\pi x}{L}\, dx$$

Taylor's Theorem (5: page 8, 8: page 10)

If u is a function of one variable which is n+1 times continuously differentiable on the interval $[x,x+h]$, then there is a $\theta \in (0,1)$ such that

$$u(x+h) = u(x) + hu'(x) + \ldots + \frac{h^n}{n!} u^{(n)}(x) + \frac{h^{n+1}}{(n+1)!} u^{(n+1)}(x+\theta h).$$

(for $h < 0$, $[x,x+h]$ is replaced by $[x+h,x]$.) If, in addition, $u^{(n+1)}$ is bounded on $[x,x+h]$, i.e.,

$$|u^{(n+1)}(\bar{x})| \leq B \qquad \bar{x} \in [x,x+h]$$

for some B, then the remainder term is of order h^{n+1}:

$$\frac{h^{n+1}}{(n+1)!} u^{(n+1)}(x+\theta h) = O(h^{n+1}) \text{ as } h \to 0.$$

Intermediate Value Theorem (8: page 10)

If a function f is differentiable on the interval [a,b], then for any real number y between f'(a) and f'(b), there is at least one point $x_0 \in [a,b]$ such that $f'(x_0) = y$.

Principle of Superposition (W: page 33)

The solution of the linear equation

$$L[u] = F$$

subject to the linear conditions

$$L_i[u] = f_i \qquad i = 1,2,\ldots,n$$

is given by

$$u = u_0 + u_1 + \ldots + u_n$$

where u_0 satisfies the problem

$$L[u_0] = F$$

$$L_i[u_0] = 0 \qquad i = 1,\ldots,n$$

and (for $k = 1,\ldots,n$) u_k satisfies the problem

$$L[u_k] = 0$$

$$L_i[u_k] = f_i\delta_{ik}$$

(δ_{ik} is the KRONECKER DELTA).

Extremum Principles for Flux (3: pages 22,23)

Let the SCALAR FIELD w satisfy $\nabla^2 w = -k$ in a domain D, with w = 0 on C (the closed boundary of D). Then if w* is any nonzero SCALAR FIELD which is differentiable on D and continuous on $D \cup C$ such that w* = 0 on C, and $\underset{\sim}{v}$* is any VECTOR FIELD such that div $\underset{\sim}{v}$* = -k in D, then

$$\frac{k\left(\int_D w^*\,dA\right)^2}{\int_D |\underset{\sim}{\text{grad}}\, w^*|^2\,dA} \le \int_D w\,dA \le \frac{1}{k}\int v^{*2}dA.$$

<u>Maximum Principle for Poisson's Equation</u> ($\underset{\sim}{W}$: pages 55,56)

If $\nabla^2 u = -F$ in D, and $F \le 0$, then

$$\max_{\underset{\sim}{r}\in D} u(\underset{\sim}{r}) \le \max_{\underset{\sim}{r}\in C} u(\underset{\sim}{r})$$

where C is the boundary of D.

<u>Finite-Difference Formulas</u>

$$u'(x) \simeq \frac{1}{h}[u(x) - u(x-h)] \qquad \text{backward-difference formula}$$

$$u'(x) \simeq \frac{1}{2h}[u(x+h) - u(x-h)] \qquad \text{central-difference formula}$$

$$u'(x) \simeq \frac{1}{h}[u(x+h) - u(x)] \qquad \text{forward-difference formula}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \simeq \frac{\delta_x^2 u}{(\Delta x)^2} + \frac{\delta_y^2 u}{(\Delta y)^2} \qquad \text{five-point formula}$$

(δ_x is the CENTRAL-DIFFERENCE OPERATOR)

$$\delta_x^2 u(x) = u(x+h) - 2u(x) + u(x-h)$$

$$= h^2 u''(x) + \frac{1}{12} h^4 u^{(4)}(x) + O(h^6).$$

The DIFFUSION OPERATOR

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

may be replaced by:

the <u>Crank-Nicolson</u> scheme

$$u_{i,j+1} - u_{i,j} = \frac{r}{2}[\delta_x^2 u_{i,j+1} + \delta_x^2 u_{i,j}],$$

the <u>Du Fort and Frankel</u> scheme

$$u_{i,j+1} - u_{i,j-1} = 2r[u_{i+1,j} - u_{i,j+1} - u_{i,j-1} + u_{i-1,j}],$$

or the explicit scheme

$$u_{i,j+1} - u_{i,j} = r\delta_x^2 u_{i,j}.$$

Here $r = k/h^2$ is the MESH RATIO, and $u_{i,j}$ denotes $u(ih,jk)$.

Courant-Friedrichs-Lewy (C.F.L.) Condition (8: page 14)

A necessary and sufficient condition for the convergence of a FINITE-DIFFERENCE SCHEME for a HYPERBOLIC equation is that the NUMERICAL DOMAIN OF DEPENDENCE of a point must include the DOMAIN OF DEPENDENCE of the same point.

Lax's Theorem (8: page 16)

Given a PROPERLY POSED, LINEAR, INITIAL VALUE PROBLEM and a linear FINITE-DIFFERENCE REPLACEMENT which is COMPATIBLE with it, the finite-difference scheme is CONVERGENT if it is STABLE.

A linear FINITE-DIFFERENCE REPLACEMENT of an INITIAL VALUE PROBLEM is STABLE if

(i) the SPECTRAL RADIUS of the matrix corresponding to the finite-difference scheme is not greater than 1 (Matrix Method)

or

(ii) the solutions to the scheme which have the form $e^{i\beta ph}\xi^q$ satisfy $|\xi| \leq 1$ (von Neumann's method).

Newton's Method

The nonlinear system of equations

$$\underset{\sim}{f}(\underset{\sim}{x}) = \underset{\sim}{0},$$

where $\underset{\sim}{f}(\underset{\sim}{x})$ denotes $\left(f_1(\underset{\sim}{x}), f_2(\underset{\sim}{x}), \ldots, f_N(\underset{\sim}{x})\right)$, may be solved by the linear ITERATIVE SCHEME

$$J_{\underset{\sim}{x}^{(n)}} \cdot (\underset{\sim}{x}^{(n)} - \underset{\sim}{x}^{(n+1)}) = \underset{\sim}{f}(\underset{\sim}{x}^{(n)})$$

where $J_{\underset{\sim}{x}^{(n)}}$ denotes the matrix whose determinant is the JACOBIAN of the functions $\{f_i\}$ at $\underset{\sim}{x}^{(n)}$.

Chain Rule

If $u, x_1, x_2, \ldots, x_N$ are variables which depend on t, and

$$u = f(x_1, \ldots, x_N)$$

then

$$\frac{du}{dt} = \sum_{i=1}^{N} \frac{\partial f}{\partial x_i} \frac{dx_i}{dt} .$$

Fourier Series

Let $\{\phi_n\}$ be an ORTHOGONAL set and f SQUARE INTEGRABLE on $[\alpha,\beta]$ with respect to the weight function ρ, and let the Fourier series for f be given by

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n .$$

Then $\sum_{n=1}^{\infty} c_n^2 \int_\alpha^\beta \phi_n^2 \rho$ converges and satisfies Bessel's Inequality

$$\sum_{n=1}^{\infty} c_n^2 \int_\alpha^\beta \phi_n^2 \rho \le \int_\alpha^\beta f^2 \rho .$$

Equality holds if and only if the Fourier series for f CONVERGES IN THE MEAN TO f, in which case we have Parseval's Equation

$$\sum_{n=1}^{\infty} c_n^2 \int_\alpha^\beta \phi_n^2 \rho = \int_\alpha^\beta f^2 \rho .$$

If $\{\phi_n\}$ is COMPLETE, and

$$f^* \sim \sum_{n=1}^{\infty} c_n^* \phi_n,$$

then

$$\sum_{n=1}^{\infty} c_n c_n^* \int_\alpha^\beta \phi_n^2 \rho = \int_\alpha^\beta ff^* \rho.$$

In the following, let L be the operator defined by

$$L : u \longmapsto (pu')' - qu,$$

and let u_k denote the kth EIGENFUNCTION of

$$Lu + \lambda\rho u = 0 \text{ in } (\alpha,\beta)$$

$$u(\alpha) = u(\beta) = 0.$$

Lagrange's Identity

$$\int (uLv - vLu) = [p(uv' - vu')]$$

over any interval in which u and v are twice CONTINUOUSLY DIFFERENTIABLE.

Theorem A

The set $\{u_k\}$ is COMPLETE for the space of functions SQUARE INTEGRABLE on (α,β) with weight function ρ.

Theorem B

If f is continuous on $[\alpha,\beta]$, $f(\alpha) = f(\beta) = 0$ and

$$\int_\alpha^\beta (pf'^2 + qf^2)$$

exists, the Fourier series of f in terms of $\{u_k\}$ is UNIFORMLY CONVERGENT to f on $[\alpha,\beta]$.

Oscillation Theorem

The eigenfunction u_k has precisely k-1 zeros in the open interval (α,β), and the zeros of successive eigenfunctions interlace.

Tests for Uniform Convergence

CAUCHY'S TEST

A necessary and sufficient condition for a sequence of functions u_n with domain I to be uniformly convergent is that given any $\varepsilon > 0$ there exists an integer N, independent of $x \in I$ such that

$$|u_n(x) - u_m(x)| < \varepsilon$$

for all $n, m > N$, and all $x \in I$.

WEIERSTRASS'S M-TEST

This is a sufficient though not necessary condition. If each term of the series

$$f(x) = \sum_{n=1}^{\infty} u_n(x) \qquad x \in I$$

is positive and

$$u_n(x) < M_n \qquad \forall x \in I,\ n \in Z^+.$$

where each M_n is independent of x, and if

$$\sum_{n=1}^{\infty} M_n$$

is convergent, then the given series for f is uniformly convergent.

Bessel Functions

J_m is a solution of BESSEL'S EQUATION OF ORDER m, and is bounded at the origin.

$$J_m(t) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{1}{2}t)^{m+2k}}{k!(m+k)!}$$

$$\frac{d}{dt}[t^{-m}J_m(t)] = -t^{-m}J_{m+1}(t)$$

$$\frac{d}{dt}[t^{m}J_m(t)] = t^{m}J_{m-1}(t)$$

$$\int_0^1 x[J_m(\sqrt{\lambda_k^{(m)}}x)]^2 dx = \tfrac{1}{2}[J_{m+1}(\sqrt{\lambda_k^{(m)}})]^2 \text{ where } J_m(\sqrt{\lambda_k^{(m)}}) = 0$$

$$J_m'(t) = \tfrac{1}{2}[J_{m-1}(t) - J_{m+1}(t)]$$

$$\frac{2t}{m} J_m(t) = J_{m+1}(t) + J_{m-1}(t)$$

4 TABLE OF INDEFINITE INTEGRALS

Function	Indefinite Integral with respect to x
$(x-a)^n$	$\frac{(x-a)^{n+1}}{n+1}$ $(n \neq -1)$
$\frac{1}{x-a}$	$\ell n\lvert x-a\rvert$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{a^2-x^2}$	$\frac{1}{2\lvert a\rvert}\ell n\left\lvert\frac{a+x}{a-x}\right\rvert$
$\frac{2x+a}{x^2+ax+b}$	$\ell n\lvert x^2+ax+b\rvert$
$\frac{2x+a}{(x^2+ax+b)^n}$	$\frac{(x^2+ax+b)^{1-n}}{1-n}$ $(n\neq 1)$
$\frac{1}{\sqrt{(a^2-x^2)}}$	$\sin^{-1}\frac{x}{\lvert a\rvert}$
$\frac{1}{\sqrt{(a^2+x^2)}}$	$\sinh^{-1}\frac{x}{\lvert a\rvert}$
$\frac{1}{\sqrt{(x^2-a^2)}}$	$\pm\cosh^{-1}\left\lvert\frac{x}{a}\right\rvert$ (sign that of x)
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\ell n\lvert\sec x\rvert$
$\cot x$	$\ell n\lvert\sin x\rvert$
$\sec x$	$\ell n\lvert\sec x + \tan x\rvert$
$\operatorname{cosec} x$	$\ell n\lvert\tan \tfrac{1}{2}x\rvert$
$\sec^2 x$	$\tan x$
$\operatorname{cosec}^2 x$	$-\cot x$
e^{ax}	$\frac{1}{a}e^{ax}$
a^x	$\frac{a^x}{\ell n\, a}$

$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\ell n(\cosh x)$
$\coth x$	$\ell n\lvert\sinh x\rvert$
$\operatorname{sech} x$	$2\tan^{-1}(e^{x})$
$\operatorname{cosech} x$	$\ell n\lvert\tanh \frac{1}{2}x\rvert$
$\operatorname{sech}^2 x$	$\tanh x$
$\operatorname{cosech}^2 x$	$-\coth x$
$\sin rx \cos nx \ (r^2 \neq n^2)$	$-\dfrac{\cos(r-n)x}{2(r-n)} - \dfrac{\cos(r+n)x}{2(r+n)}$
$\sin rx \sin nx \ (r^2 \neq n^2)$	$\dfrac{\sin(r-n)x}{2(r-n)} - \dfrac{\sin(r+n)x}{2(r+n)}$
$\cos rx \cos nx \ (r^2 \neq n^2)$	$\dfrac{\sin(r-n)x}{2(r-n)} + \dfrac{\sin(r+n)x}{2(r+n)}$
$e^{ax}\cos(bx + c)$	$\dfrac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)]$
$e^{ax}\sin(bx + c)$	$\dfrac{e^{ax}}{a^2+b^2}[a\sin(bx+c) - b\cos(bx+c)]$

5 TABLE OF DEFINITE INTEGRALS

In the following m and n are integers.

$$\int_0^\pi \sin mx \sin nx \, dx = 0 \qquad m \neq n$$

$$\int_0^\pi \sin nx dx = \frac{\pi}{2} \qquad n \neq 0$$

$$\int_0^\pi \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \frac{1}{2}\pi & m = n \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx = 0$$

$$\int_{-\pi}^{\pi} \cos nx \, dx = \begin{cases} 0 & n \neq 0 \\ 2\pi & n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$